

# Discrete Flavour Symmetries in Light of T2K

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## Abstract

We show that a non-vanishing angle  $\theta_{13}$  of order 0.1 can be predicted in the framework of discrete flavour symmetries. We assume that left-handed leptons transform as triplets under a group  $G_f$  which is broken in such a way that neutrino and charged lepton sectors remain invariant under the subgroups  $G_\nu$  and  $G_e$  of  $G_f$ , respectively. In this limit mixing angles and the Dirac  $CP$  violating phase  $\delta_{CP}$  are determined. By choosing  $G_f = \Delta(6n^2)$  ( $n = 4, 8$ ),  $G_\nu = Z_2 \times Z_2$  and  $G_e = Z_3$  we find  $\sin^2 \theta_{13} = 0.045(0.011)$  for  $n = 4(8)$ . At the same time  $\theta_{23}$  and  $\theta_{12}$  remain close to their experimental best fit values, particularly in the case  $n = 8$ , where  $\sin^2 \theta_{23} \approx 0.424$  and  $\sin^2 \theta_{12} \approx 0.337$ .  $\delta_{CP}$  is predicted to be 0 or  $\pi$  so that  $CP$  is conserved in our examples.

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# 1 Introduction

Neutrino oscillation experiments, interpreted in the framework of three active neutrino species, have shown that the mixing angle  $\theta_{13}$  is considerably smaller than the other two. Until recently  $\theta_{13}$  was actually compatible with being zero at the  $2\sigma$  level. In June 2011 the T2K collaboration reported indication of electron neutrino appearance from a muon neutrino beam of energy about 0.6 GeV produced at J-PARC, 295 km away from the detector [1]. This excludes the hypothesis  $\theta_{13} = 0$  at the level of  $2.5\sigma$  and favors  $\theta_{13}$  around  $0.17 \div 0.19$ <sup>1</sup>, not far from the upper limits set by CHOOZ [2] and by MINOS [3]. By itself such an indication is not conclusive but it adds in an interesting way to other previous hints suggesting a non-vanishing reactor mixing angle in that range. In particular a tension between the values of the oscillation parameters extracted from KamLAND and from solar neutrinos is alleviated for  $\theta_{13} \approx 0.1$  [4]. A recent global analysis of the data [5] provides evidence for nonzero  $\theta_{13}$  at the  $3\sigma$  level

$$\sin^2 \theta_{13} = 0.021(0.025) \pm 0.007 \quad (1\sigma) \quad . \quad (1)$$

The central value 0.15 (0.16) of  $\theta_{13}$  depends on the assumed reactor antineutrino flux, with the results from the new flux estimate [6] shown in parenthesis. The two other angles are (at  $1\sigma$  level)

$$\sin^2 \theta_{23} = 0.42^{+0.08}_{-0.03} \quad , \quad \sin^2 \theta_{12} = 0.306(0.312)^{+0.018(0.017)}_{-0.015(0.016)} \quad . \quad (2)$$

Notice that the difference between new and old fluxes is only relevant for  $\theta_{12}$  and  $\theta_{13}$ .

Finding a consistent and economic explanation of fermion masses and mixing angles is one of the main open problems in particle physics today. Given the key role traditionally played by symmetries in understanding particle properties, flavour symmetries have captured considerable attention in this context. The peculiar mixing in the lepton sector, with two large angles and a small one, significantly close to simple patterns such as  $\sin^2 \theta_{23} = 1/2$ ,  $\sin^2 \theta_{12} = 1/3$  and  $\sin^2 \theta_{13} = 0$ , has revived the interest in discrete groups, that naturally incorporate this kind of patterns. The mentioned pattern is called tribi-maximal (TB) mixing [7] and has received much attention in the last years. Many efforts have been made to reproduce it in concrete models and one of the simplest ways is via a non-trivially broken flavour symmetry based on a small discrete group, such as  $A_4$  and  $S_4$ , respectively, see [8] for reviews. TB mixing is in general obtained in a certain limit of the theory in which corrections are neglected. Deviations thereof are expected to arise from several sources and non-vanishing  $\theta_{13}$  is predicted. However, these corrections tend to affect all mixing angles by a similar amount, and, given the good agreement between the predicted and the observed value of the solar mixing angle, only small corrections, up to 0.03, are admissible. As a consequence,  $\sin \theta_{13}$  is also expected to be of order 0.03 and thus not compatible with the result in eq. (1). The same arguments apply to other mixing patterns that have been derived from a non-trivial breaking of other discrete groups, such as patterns with  $\theta_{12}$  given in terms of the golden ratio [9] or given as  $\sin^2 \theta_{12} = 1/4$  [10].

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<sup>1</sup>Angles are given in radian.

Also in these cases the good agreement between predicted and observed value of the solar angle suggests that corrections should be small, of the order of few percent.

Several attempts have been made to explain largish  $\theta_{13}$  in a framework predicting TB mixing at leading order (LO): the introduction of corrections leaving one row or column of the mixing matrix unchanged [11, 12]; the addition of scalars transforming as non-trivial singlets in  $A_4$  models [13] leading to an explicit breaking of the  $\mu\tau$  exchange symmetry of the neutrino sector (in the charged lepton mass basis)<sup>2</sup> or the assumption that the different symmetry breaking parameters associated with the neutrino and charged lepton sectors, respectively, are significantly different in size [15]. In other models [16] bimaximal mixing is derived as mixing pattern at LO which requires sizable corrections to the solar mixing angle and easily leads to large  $\theta_{13}$  as well. However, in this case the atmospheric mixing angle has to be protected from too large corrections and it becomes difficult to precisely predict a particular value for the mixing angles.

In the present note we change perspective and show that it is indeed possible to derive mixing patterns with  $\theta_{13} \neq 0$  at LO from discrete flavour symmetries  $G_f$ . We break  $G_f$  in a non-trivial way so that the neutrino and the charged lepton sectors are (separately) invariant under two different subgroups of the original group  $G_f$ . We show two examples, based on the flavour groups  $\Delta(6n^2)$ , with  $n = 4$  and  $n = 8$  respectively, in which not only  $\theta_{13}$  of the correct size is predicted, but also  $\theta_{23}$  and  $\theta_{12}$  are close to their experimental best fit values, e.g. for  $n = 8$  we get  $\sin^2 \theta_{23} \approx 0.424$  and  $\sin^2 \theta_{12} \approx 0.337$ .

## 2 Framework

In our approach the theory is invariant under a discrete flavour group  $G_f$  under which the three generations of  $SU(2)_L$  lepton doublets  $l$  transform as a faithful three-dimensional irreducible representation. The group  $G_f$  can be a symmetry of the full Lagrangian or just an accidental one arising in some LO approximation, for instance by neglecting operators of high dimensionality. The lepton mixing matrix  $U_{PMNS}$  is determined by the residual symmetries of the neutrino and the charged lepton sectors. Indeed, a crucial assumption is that the neutrino mass matrix  $m_\nu$  and the combination  $m_e^\dagger m_e$ ,  $m_e$  being the charged lepton mass matrix<sup>3</sup>, are separately invariant under the subgroups  $G_\nu$  and  $G_e$  of  $G_f$ , respectively. We analyse possible mixing patterns independently from a specific model realisation, and therefore we do neither specify the details of the symmetry breaking mechanism nor the transformation properties of fields under  $G_f$  other than  $l \sim 3$ . Neutrinos are assumed to be Majorana particles, which fixes  $G_\nu$ . With a single generation, the only transformation of a Majorana neutrino leaving invariant its mass term is a change of sign. If three generations are present, it can be shown [17] that the appropriate invariance group of the neutrino sector is the product of two commuting parities, the Klein group  $Z_2 \times Z_2$ , allowing for an independent relative change of sign of any neutrino. We assume  $G_e$  to be abelian, since non-abelian subgroups would result in a complete or partial degeneracy of the mass spectrum, a feature difficult to reconcile with the observed charged lepton mass hierarchy.

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<sup>2</sup>For a similar model with the flavour symmetry  $S_4$ , see [14].

<sup>3</sup>In our convention  $SU(2)_L$  doublets are on the right of  $m_e$ .

We choose  $G_e = Z_3$  in our examples. This is actually the minimal choice of  $G_e$  that can ensure three independent mass parameters and allows to uniquely fix the mixing angles in the charged lepton sector, up to permutations. Since we are interested in minimal realisations we require that the generators of the subgroups  $G_\nu$  and  $G_e$  give rise to the whole group  $G_f$  and not only a subgroup of it, which could otherwise be used as starting point instead of  $G_f$ .

We call  $\rho$  the three-dimensional representation of  $G_f$  for the lepton doublets  $l$  and the elements  $g_{\nu i}$  of  $G_\nu$  and  $g_{ei}$  of  $G_e$  are given by matrices  $\rho(g_{\nu i})$  and  $\rho(g_{ei})$ , respectively. The invariance requirements read

$$\rho(g_{\nu i})^T m_\nu \rho(g_{\nu i}) = m_\nu \quad \text{and} \quad \rho(g_{ei})^\dagger m_e^\dagger m_e \rho(g_{ei}) = m_e^\dagger m_e \quad . \quad (3)$$

Since  $\rho$  is a unitary representation and  $G_\nu$  and  $G_e$  are abelian, there exist two unitary transformations  $\Omega_\nu$  and  $\Omega_e$  that diagonalise the matrices  $\rho(g_{\nu i})$  and  $\rho(g_{ei})$

$$\rho(g_{\nu i})_{diag} = \Omega_\nu^\dagger \rho(g_{\nu i}) \Omega_\nu \quad \text{and} \quad \rho(g_{ei})_{diag} = \Omega_e^\dagger \rho(g_{ei}) \Omega_e \quad . \quad (4)$$

Requiring eq.(3) to be fulfilled has as consequence that  $\Omega_\nu$  and  $\Omega_e$  are also the transformations that diagonalise  $m_\nu$  and  $m_e^\dagger m_e$ , respectively. It follows that the lepton mixing matrix is

$$U_{PMNS} = \Omega_e^\dagger \Omega_\nu \quad , \quad (5)$$

up to some redefinitions. Indeed  $\Omega_e$  and  $\Omega_\nu$  are defined up to a multiplication from the right by a diagonal matrix  $K_{e,\nu}$  of phases,

$$\Omega_e \rightarrow \Omega_e K_e \quad \text{and} \quad \Omega_\nu \rightarrow \Omega_\nu K_\nu \quad . \quad (6)$$

The phase freedom associated with  $K_e$  can be used to remove three phases from the combination  $\Omega_e^\dagger \Omega_\nu$ , while the phase freedom associated with  $K_\nu$  can be employed to get real and positive eigenvalues of  $m_\nu$ . After that we are left with three physical phases in  $\Omega_e^\dagger \Omega_\nu$ : the Dirac  $CP$  phase  $\delta_{CP}$  and the two Majorana phases. The latter cannot be predicted in our approach since the eigenvalues of  $m_\nu$  remain unconstrained by the requirement in eq.(3). The Dirac phase is instead determined by  $\Omega_e^\dagger \Omega_\nu$ . Similarly to the neutrino masses also the charged lepton masses remain free parameters and thus we cannot fix the ordering of both rows and columns of  $U_{PMNS}$ . We use this freedom by choosing the order that allows mixing angles as close as possible to the experimental best fit values. Note that also the exact value of  $\delta_{CP}$  depends on the actual ordering of rows and columns and thus we can determine its value only up to  $\pi$ .

From eqs. (3,4) we see that the mixing matrix  $U_{PMNS}$  is not sensitive to the overall sign of the matrices representing the elements of  $G_\nu$  and  $G_e$ . Moreover if we replace the matrices representing the elements of  $G_\nu$  and  $G_e$  by their complex conjugates, the mixing matrix  $U_{PMNS}$  becomes complex conjugated as well. Therefore representations  $\rho$  and  $\rho'$  that differ by an overall sign in the elements  $g_{\nu i}$  and  $g_{ei}$  and/or that are related by a complex conjugation are not discussed separately.

Finally, concerning the choice of the flavour group  $G_f$ , we consider two examples in which  $G_f$  is  $\Delta(96)$  and  $\Delta(384)$ , respectively.

Summarising, in our approach the lepton mixing originates from the misalignment of the remnant subgroups in neutrino and charged lepton sectors. With the knowledge of  $G_\nu$  and  $G_e$  mixing angles and the phase  $\delta_{CP}$  are predicted, while lepton masses and Majorana phases remain unconstrained.

### 3 Mixing patterns with non-vanishing $\theta_{13}$

We present two examples in which the lepton mixing matrix has non-vanishing  $\theta_{13}$  and is determined as outlined above. These examples are based on the two flavour groups  $G_f = \Delta(96)$  and  $G_f = \Delta(384)$ , respectively. They belong to the series  $\Delta(6n^2)$ ,  $n$  being a natural number, and are subgroups of the (inhomogeneous) modular group  $\Gamma$  which is isomorphic to the projective special linear group  $PSL(2, Z)$ . We note that the group  $S_4$  with which TB mixing can be predicted is isomorphic to  $\Delta(24)$ .

In our first example  $G_f = \Delta(96)$  the generators  $S$  and  $T$  fulfill the relations [18]<sup>4</sup>

$$S^2 = (ST)^3 = T^8 = \mathbb{1} \quad , \quad (ST^{-1}ST)^3 = \mathbb{1} \quad , \quad (7)$$

The group has ten conjugacy classes:  $\{E\}$ ,  $3C_2$ ,  $12C_2$ ,  $32C_3$ ,  $3C_4$ ,  $3C'_4$ ,  $6C_4$ ,  $12C_4$ ,  $12C_8$  and  $12C'_8$ , where  $E$  is the identity, the first number stands for the number of elements in the class and the index denotes the order of the elements. The group has 96 elements which are of order 1, 2, 3, 4 and 8, respectively. There are ten irreducible representations: two singlets, one doublet, six triplets and one sextet. The character table of  $\Delta(96)$  can be found in [18]. Two of the irreducible triplets are not faithful representations and are not used in our analysis. The remaining four triplets  $\rho_i$  ( $i = 1...4$ ) are related among each other either by an overall change of sign of both  $\rho_i(S)$  and  $\rho_i(T)$  and/or through complex conjugation. For this reason we restrict our analysis to a particular three-dimensional representation with

$$\rho(S) = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \quad \rho(T) = \begin{pmatrix} e^{\frac{6\pi i}{4}} & 0 & 0 \\ 0 & e^{\frac{7\pi i}{4}} & 0 \\ 0 & 0 & e^{\frac{3\pi i}{4}} \end{pmatrix} \quad . \quad (8)$$

By choosing  $G_\nu = Z_2 \times Z_2$  and  $G_e = Z_3$ , we find seven distinct subgroups  $Z_2 \times Z_2$  and sixteen  $Z_3$  subgroups, giving rise to 112 different possibilities for the lepton mixing matrix. Requiring that we generate the original group  $\Delta(96)$  with the generators of  $G_\nu$  and  $G_e$  leaves us with 48 different combinations. As can be shown, all these are related by group transformations and thus produce the same  $U_{PMNS}$ , up to permutations of rows and columns and phase redefinitions.

A possible choice for the generators of  $G_\nu = Z_2 \times Z_2$  and  $G_e = Z_3$  is given by

$$\begin{aligned} G_\nu : & \quad \{S, ST^4 ST^4\} \\ G_e : & \quad ST \end{aligned} \quad . \quad (9)$$

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<sup>4</sup>In the Appendix we discuss the relation between  $S$  and  $T$  and the generators  $a$ ,  $b$ ,  $c$  and  $d$  chosen in [18] to define the groups  $\Delta(6n^2)$ .

The absolute values  $||U_{PMNS}||$  of the mixing matrix are

$$||U_{PMNS}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}(\sqrt{3}+1) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ \frac{1}{2}(\sqrt{3}-1) & 1 & \frac{1}{2}(\sqrt{3}+1) \\ 1 & 1 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.789 & 0.577 & 0.211 \\ 0.211 & 0.577 & 0.789 \\ 0.577 & 0.577 & 0.577 \end{pmatrix} . \quad (10)$$

With the ordering chosen in eq. (10), the mixing angles and the Dirac  $CP$  phase read <sup>5</sup>

$$\begin{aligned} \sin^2 \theta_{23} &= \frac{5+2\sqrt{3}}{13} \approx 0.651 \\ \sin^2 \theta_{12} &= \frac{8-2\sqrt{3}}{13} \approx 0.349 \\ \sin^2 \theta_{13} &= \frac{2-\sqrt{3}}{6} \approx 0.045 \\ \delta_{CP} &= \pi . \end{aligned} \quad (\text{M1}) \quad (11)$$

If we exchange the second and third rows in  $U_{PMNS}$  we have

$$\begin{aligned} \sin^2 \theta_{23} &= \sin^2 \theta_{12} = \frac{8-2\sqrt{3}}{13} \approx 0.349 \\ \sin^2 \theta_{13} &= \frac{2-\sqrt{3}}{6} \approx 0.045 \\ \delta_{CP} &= 0 . \end{aligned} \quad (\text{M2}) \quad (12)$$

It is interesting to note that  $CP$  is conserved in both cases. The patterns M1 and M2 give rise to mixing angles which are compatible with the present data, however only at roughly the  $3\sigma$  level, as shown in figure 1.

In the second example  $G_f = \Delta(384)$   $S$  and  $T$  fulfill (see footnote 3)

$$S^2 = (ST)^3 = T^{16} = \mathbb{1} , \quad (ST^{-1}ST)^3 = \mathbb{1} . \quad (13)$$

The conjugacy classes are 24:  $\{E\}$ ,  $3C_2$ ,  $24C_2$ ,  $128C_3$ ,  $3C_4$ ,  $3C'_4$ ,  $6C_4$ ,  $24C_4$ ,  $3C_8^i$ ,  $6C_8^j$ ,  $24C_8^k$ ,  $24C_{16}^i$ , ( $i = 1\dots 4$ ), ( $j = 1\dots 6$ ), ( $k = 1, 2$ ). The 384 elements of this group are of order 1, 2, 3, 4, 8 and 16, respectively. There are 24 irreducible representations: two singlets, one doublet, 14 triplets and seven sextets. Six triplets are unfaithful representations and are not considered here. The remaining eight triplets can be divided into two sets containing four triplets each whose matrices  $\rho(S)$  and  $\rho(T)$  for the generators  $S$  and  $T$  are related by an overall change of sign and/or through complex conjugation as in the previous example. As a consequence, we only need to consider the following two irreducible three-dimensional representations

$$\rho_1(S) = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \quad \rho_1(T) = \begin{pmatrix} e^{\frac{14\pi i}{8}} & 0 & 0 \\ 0 & e^{\frac{5\pi i}{8}} & 0 \\ 0 & 0 & e^{\frac{13\pi i}{8}} \end{pmatrix} , \quad (14)$$

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<sup>5</sup>We use the conventions of [19].

and

$$\rho_2(S) = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \quad \rho_2(T) = \begin{pmatrix} e^{\frac{6\pi i}{8}} & 0 & 0 \\ 0 & e^{\frac{9\pi i}{8}} & 0 \\ 0 & 0 & e^{\frac{\pi i}{8}} \end{pmatrix} . \quad (15)$$

By choosing  $G_\nu = Z_2 \times Z_2$  and  $G_e = Z_3$ , we find 13 distinct subgroups  $Z_2 \times Z_2$  and 64  $Z_3$  subgroups, resulting in 832 different possibilities. Again, considering only those cases in which the generators of  $G_\nu$  and  $G_e$  give rise to the original group  $\Delta(384)$ , we are left with 384 combinations. It is easy to check that all these combinations are related by group transformations and thus necessarily the same mixing matrix  $U_{PMNS}$  is obtained, up to phase redefinitions and permutations of rows and columns. These statements hold for both representations  $\rho_1$  and  $\rho_2$  and moreover both of them give rise to the same mixing pattern.

A possible choice for the generators of  $G_\nu = Z_2 \times Z_2$  and  $G_e = Z_3$  is given by

$$\begin{aligned} G_\nu : & \quad \{S, ST^8 ST^8\} \\ G_e : & \quad ST \end{aligned} . \quad (16)$$

The absolute values  $||U_{PMNS}||$  of the mixing matrix are

$$\begin{aligned} ||U_{PMNS}|| &= \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\sqrt{4+\sqrt{2}+\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}-\sqrt{6}} \\ \frac{1}{2}\sqrt{4+\sqrt{2}-\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}+\sqrt{6}} \\ \sqrt{1-\frac{1}{\sqrt{2}}} & 1 & \sqrt{1+\frac{1}{\sqrt{2}}} \end{pmatrix} \\ &\approx \begin{pmatrix} 0.810 & 0.577 & 0.107 \\ 0.497 & 0.577 & 0.648 \\ 0.312 & 0.577 & 0.754 \end{pmatrix} . \end{aligned} \quad (17)$$

With the ordering chosen in eq. (17), the mixing angles and the Dirac  $CP$  phase are

$$\begin{aligned} \sin^2 \theta_{23} &= \frac{4 - \sqrt{2} + \sqrt{6}}{8 + \sqrt{2} + \sqrt{6}} \approx 0.424 \\ \sin^2 \theta_{12} &= \frac{4}{8 + \sqrt{2} + \sqrt{6}} \approx 0.337 \\ \sin^2 \theta_{13} &= \frac{4 - \sqrt{2} - \sqrt{6}}{12} \approx 0.011 \\ \delta_{CP} &= 0 . \end{aligned} \quad (18)$$

If we exchange the second and third rows in  $U_{PMNS}$  we have

$$\begin{aligned} \sin^2 \theta_{23} &= \frac{4 + 2\sqrt{2}}{8 + \sqrt{2} + \sqrt{6}} \approx 0.576 \\ \sin^2 \theta_{12} &= \frac{4}{8 + \sqrt{2} + \sqrt{6}} \approx 0.337 \\ \sin^2 \theta_{13} &= \frac{4 - \sqrt{2} - \sqrt{6}}{12} \approx 0.011 \\ \delta_{CP} &= \pi . \end{aligned} \quad (19)$$

We observe that  $CP$  is conserved in both cases. The mixing pattern **M3** is compatible with the present data at the  $2\sigma$  level, as can be seen from figure 1, and provides an excellent first order approximation in a theoretical description of the observed lepton mixing angles.

Notice that by taking  $G_f = \Delta(24) \simeq S_4$  and by choosing  $G_\nu = Z_2 \times Z_2$  and  $G_e = Z_3$  such that  $G_f$  is generated by the elements of  $G_\nu$  and  $G_e$ , the unique mixing pattern achieved with our approach is TB mixing, see also [17].

It is interesting to note that both mixing matrices  $U_{PMNS}$  whose absolute values are displayed in eqs. (10) and (17) can be brought into a form in which the second column has three entries equal to  $1/\sqrt{3}$ . In doing so it becomes obvious that the results presented are related in a particular way to the TB mixing matrix whose entries of the second column are usually defined to be all equal to  $1/\sqrt{3}$  as well. Indeed, the TB mixing matrix

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (20)$$

can be modified by a rotation in the 13 plane acting from the right

$$U_{PMNS} = U_{TB} U_{13}(\alpha) \quad \text{with} \quad U_{13}(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix} . \quad (21)$$

It is immediate to show that, by taking  $\alpha = -\pi/12$  and  $\alpha = \pi/24$ , the resulting mixing matrices are identical, in absolute value, to the matrices in eqs. (10) and (17), respectively. Taking the opposite signs,  $\alpha = \pi/12$  and  $\alpha = -\pi/24$ , we get the matrices with absolute values of the same form as in eqs. (10) and (17), respectively, with second and third rows exchanged. Such perturbations from TB mixing with  $\alpha$  arbitrary have been already discussed in the literature [11, 12, 15]. For generic  $\alpha$ , the mixing angles read

$$\sin^2 \theta_{12} = \frac{1}{2 + \cos 2\alpha} , \quad \sin^2 \theta_{23} = \frac{1}{2} - \frac{\sqrt{3} \sin 2\alpha}{4 + 2 \cos 2\alpha} , \quad \sin^2 \theta_{13} = \frac{2}{3} \sin^2 \alpha . \quad (22)$$

For small  $\alpha$ , we can expand the results

$$\sin^2 \theta_{12} \approx \frac{1}{3} + \frac{2\alpha^2}{9} , \quad \sin^2 \theta_{23} \approx \frac{1}{2} - \frac{\alpha}{\sqrt{3}} , \quad \sin^2 \theta_{13} \approx \frac{2\alpha^2}{3} \quad (23)$$

showing that the deviation from the value of TB mixing of  $\sin^2 \theta_{12}$ , the best measured quantity among the three mixing angles, is quadratic in  $\alpha$ , whereas the leading correction to  $\sin^2 \theta_{23} = 1/2$  is linear in  $\alpha$ .

## 4 Conclusions

Recent results of the T2K experiment and of a global fit of the neutrino oscillation data point to non-vanishing  $\theta_{13}$  at the  $3\sigma$  level. The best fit value of  $\theta_{13}$  is around  $0.15 \div 0.16$ , smaller than the ones of the other angles, but much larger than 0.02, the  $1\sigma$  experimental



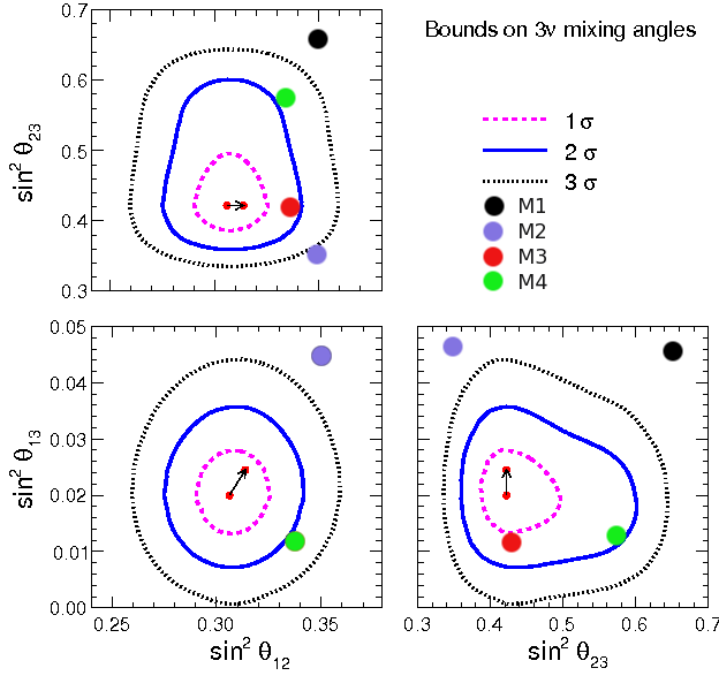


Figure 1: Values of  $\sin^2 \theta_{ij}$  for the four different mixing patterns M1 (black), M2 (violet), M3 (red) and M4 (green). The contours show the  $1\sigma$  (pink dashed line),  $2\sigma$  (blue solid line) and  $3\sigma$  (black dotted line) levels and are taken from [5]. The small dots indicate the best fit values of the mixing angles and the arrows the effect of the new estimates of the reactor antineutrino flux. Note that in the  $\sin^2 \theta_{12}$ - $\sin^2 \theta_{13}$  plane the points of M1 and M2 as well as of M3 and M4 lie on top of each other, since they only differ in the value of  $\sin^2 \theta_{23}$ .

error on the solar angle  $\theta_{12}$ . If future data confirm this result, many models giving rise at LO to mixing patterns with vanishing  $\theta_{13}$ , such as TB mixing, become disfavoured, because corrections, expected in these models, generically lead to too small  $\theta_{13}$ . A particular elegant mechanism to produce simple mixing patterns is based on discrete flavour symmetries. The latter are broken in a non-trivial way and as a consequence give rise to mixing angles whose values only depend on the properties of the flavour symmetry, but not on lepton masses. After the T2K data the natural question is whether such symmetries still remain a valuable tool to describe flavour mixing. One obvious possibility is to modify the existing models which lead to  $\theta_{13} = 0$  at LO, by means of suitable perturbations to match the experimental data.

In this note we have shown that it is possible to predict a small, non-vanishing  $\theta_{13}$  even in the absence of such perturbations, in the framework of non-trivially broken discrete symmetries. The theory is invariant under a discrete flavour group  $G_f$ , broken in such a way that the relevant mass matrices  $m_\nu$  and  $m_e^\dagger m_e$  have a residual invariance under

the subgroups  $G_\nu$  and  $G_e$ , respectively. The lepton mixing matrix originates from the mismatch of these two subgroups and from their specific embedding into  $G_f$ . By choosing  $G_f = \Delta(6n^2)$  ( $n = 4, 8$ ),  $G_\nu = Z_2 \times Z_2$ , and  $G_e = Z_3$  we find  $\sin^2 \theta_{13} = 0.045(0.011)$  for  $n = 4(8)$ . At the same time  $\theta_{23}$  and  $\theta_{12}$  are close to their experimental best fit values, especially in the case  $n = 8$  in which we find  $\sin^2 \theta_{23} \approx 0.424$  and  $\sin^2 \theta_{12} \approx 0.337$ , see mixing pattern M3. The  $CP$  violating phase  $\delta_{CP}$  is predicted to be 0 or  $\pi$  so that  $CP$  is conserved, at LO. Our proposed mixing patterns are related to TB mixing in a simple way, namely they can be obtained through a rotation by an angle  $\alpha$ ,  $\alpha = \pm\pi/12$  for  $n = 4$  and  $\alpha = \pm\pi/24$  for  $n = 8$ , respectively, in the 13 plane acting from the right on the TB mixing matrix.

Finally, we would like to mention that the presented results are part of a systematic investigation of finite subgroups of  $PSL(2, Z)$ , and that a comprehensive study is detailed in a future publication.

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## Appendix

The groups  $\Delta(6n^2)$  are non-abelian finite subgroups of  $SU(3)$  of order  $6n^2$ . They are isomorphic to the semidirect product of  $S_3$ , the smallest non-abelian finite group, with  $Z_n \times Z_n$  [18, 20],

$$\Delta(6n^2) \simeq (Z_n \times Z_n) \rtimes S_3 \quad . \quad (24)$$

They can be defined in terms of four generators  $a, b, c, d$ , satisfying

$$a^3 = b^2 = (ab)^2 = c^n = d^n = \mathbb{1}, \quad (25)$$

$$cd = dc, \quad (26)$$

$$\begin{aligned} aca^{-1} &= c^{-1}d^{-1}, & ada^{-1} &= c, \\ bcb^{-1} &= d^{-1}, & bdb^{-1} &= c^{-1}. \end{aligned} \quad (27)$$

The elements  $a$  and  $b$  are the generators of  $S_3$  while  $c$  and  $d$  generate  $Z_n \times Z_n$ . Here we show that the relations, found in eqs. (7) and (13) for  $n = 4$  and  $n = 8$  and given in terms of only two generators  $S$  and  $T$ , indeed define the same group as those given for  $a, b, c$  and  $d$ . In the case of  $n = 4$ , i.e.  $\Delta(96)$ ,  $S$  and  $T$  are related to the generators above through

$$\begin{aligned} a &= T^5 S T^4 & b &= S T^2 S T^5 \\ c &= S T^2 S T^4 & d &= S T^2 S T^6 \end{aligned} \quad . \quad (28)$$

For  $\Delta(384)$  the relation is

$$\begin{aligned} a &= T^{15}ST^8 & b &= ST^6ST^3 \\ c &= ST^2ST^4 & d &= ST^2ST^{14} \end{aligned} \quad . \quad (29)$$

Note that if we apply the same similarity transformation to the elements  $X_i$  given on the right-hand side of the equations for  $a, b, c, d$

$$X_i \rightarrow g X_i g^{-1} \quad (30)$$

with  $g$  being an element of the group, we obtain an equally valid realisation of  $a, b, c, d$ .

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